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# Lie-algebraic approach for pricing moving barrier options with time-dependent parameters ☆

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## Abstract

In this paper we apply the Lie-algebraic technique for the valuation of moving barrier options with time-dependent parameters. The value of the underlying asset is assumed to follow the constant elasticity of variance (CEV) process. By exploiting the dynamical symmetry of the pricing partial differential equations, the new approach enables us to derive the analytical kernels of the pricing formulae straightforwardly, and thus provides an efficient way for computing the prices of the moving barrier options. The method is also able to provide tight upper and lower bounds for the exact prices of CEV barrier options with fixed barriers. In view of the CEV model being empirically considered to be a better candidate in equity option pricing than the traditional Black–Scholes model, our new approach could facilitate more efficient comparative pricing and precise risk management in equity derivatives with barriers by incorporating term-structures of interest rates, volatility and dividend into the CEV option valuation model.



*Keywords:* Options; Constant elasticity of variance; Partial differential equation; Lie algebra

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## 1. Introduction

Recently Lo and Hui [1,2] introduced a Lie-algebraic method to the field of finance for the pricing of financial derivatives with time-dependent model parameters. This new method is based upon the Wei–Norman theorem [3] and has never been used in the field of finance. By exploiting the well-defined algebraic structures of the pricing partial differential equations, analytical closed-form pricing formulae can be derived for financial derivatives with time-dependent parameters. It is the purpose of this communication to extend the Lie-algebraic approach to the valuation of moving barrier options with time-dependent parameters. In the valuation of these moving barrier options the value of the underlying asset is assumed to follow the constant elasticity of variance (CEV) diffusion process:

$$dS = \mu(t) S\,dt + \sigma(t) S^{\beta/2}\,dZ, \quad 0 \leqslant \beta < 2, \tag{1}$$

where $\mu$ is the instantaneous mean, $\sigma S^{\beta/2}$ is the instantaneous variance of the stock price, $dZ$ is a Weiner process and $\beta$ is the elasticity factor. The equation shows that the instantaneous variance of the percentage price change is equal to $\sigma^2/S^{2-\beta}$ and is a direct inverse function of the stock price. In the limiting case $\beta = 2$, the CEV model returns to the conventional Black–Scholes model in which the variance rate is independent of the stock price. In another case $\beta = 0$, it is the Ornstein–Uhlenbeck model. In this paper we generalize the Lie-algebraic technique to derive the analytical kernels of the pricing formulae of the moving barrier options with time-dependent parameters, and thus provide an efficient way for computing the prices of both up-and-out and down-and-out barrier options (both call and put options). Furthermore, making use of the maximum principle for the parabolic partial differential equation [4], our approach can be applied to yield very tight upper and lower bounds of the exact prices of CEV barrier options with fixed barriers.

The remainder of this paper is organized as follows. Section 2 outlines the Wei–Norman theorem and its applications. Section 3 applies the Lie-algebraic technique to the valuation problem of the CEV options with time-dependent parameters. Section 4 presents the derivation of the analytical kernels of the pricing formulae of the moving barrier options. We also illustrate how the results can be applied to compute the upper and lower bounds of the exact prices of CEV barrier options with fixed barriers. Finally, Section 5 briefly concludes the paper.

## 2. Wei–Norman theorem

Consider the linear operator differential equation of the first order

$$\frac{dU(t)}{dt} = H(t)U(t), \qquad U(0) = 1, \tag{2}$$

where $H$ and $U$ are both time-dependent linear operators in a Banach space or a finite-dimensional space. According to the Wei–Norman theorem [3], if the operator $H$ can be expressed as

$$H(t) = \sum_{n=1}^{N} a_n(t) L_n, \tag{3}$$

where $a_n$'s are scalar functions of time and $L_n$ are the generators of an $N$-dimensional solvable Lie algebra or the real split 3-dimensional simple Lie algebra, then the operator $U$ can assume the following form:

$$U(t) = \prod_{n=1}^{N} \exp\bigl[g_n(t)L_n\bigr]. \tag{4}$$

Here the $g_n$'s are time-dependent scalar functions to be determined. To find the $g_n$'s, we simply substitute Eqs. (2) and (3) into Eq. (1), and compare the two sides term by term to obtain a set of coupled nonlinear differential equations

$$\frac{dg_n(t)}{dt} = \sum_{m=1}^{N} \eta_{nm} a_m(t), \qquad g_n(0) = 0, \tag{5}$$

where $\eta_{nm}$ are nonlinear functions of $g_n$'s. Thus, we have transformed the linear operator differential equation in Eq. (2) to a set of coupled nonlinear differential equations of scalar functions in Eq. (5).

For illustration, we consider the special case that the generators $L_n$'s form the Heisenberg–Weyl Lie algebra defined by the commutation relations:

$$[L_1, L_2] = L_3, \qquad [L_1, L_3] = [L_2, L_3] = 0. \tag{6}$$

Then $H$ is given by

$$H(t) = a_1(t)L_1 + a_2(t)L_2 + a_3(t)L_3. \tag{7}$$

According to the Wei–Norman theorem, $U(t)$ can be expressed as

$$U(t) = \exp\bigl[g_1(t)L_1\bigr]\exp\bigl[g_2(t)L_2\bigr]\exp\bigl[g_3(t)L_3\bigr]. \tag{8}$$

By differentiation, we obtain

$$\begin{aligned}
\frac{dU(t)}{dt}U(t)^{-1} &= \frac{dg_1(t)}{dt}L_1 + \frac{dg_2(t)}{dt}\exp\bigl[g_1(t)L_1\bigr]L_2\exp\bigl[-g_1(t)L_1\bigr] \\
&\quad + \frac{dg_3(t)}{dt}\exp\bigl[g_1(t)L_1\bigr]\exp\bigl[g_2(t)L_2\bigr]L_3\exp\bigl[-g_2(t)L_2\bigr]\exp\bigl[-g_1(t)L_1\bigr] \\
&= \frac{dg_1(t)}{dt}L_1 + \frac{dg_2(t)}{dt}L_2 + \left[\frac{dg_3(t)}{dt} + g_1(t)\frac{dg_2(t)}{dt}\right]L_3.
\end{aligned} \tag{9}$$

Comparing Eqs. (6) and (8) gives a set of three coupled nonlinear differential equations:

$$\begin{aligned}
&\frac{dg_1(t)}{dt} = a_1(t), \\
&\frac{dg_2(t)}{dt} = a_2(t), \\
&\frac{dg_3(t)}{dt} + g_1(t)\frac{dg_2(t)}{dt} = a_3(t).
\end{aligned} \tag{10}$$

It is not difficult to show that the set of differential equations can be easily solved by quadrature:

$$g_1(t) = \int_0^t d\tau\, a_1(\tau),$$

$$g_2(t) = \int_0^t d\tau\, a_2(\tau),$$

$$g_3(t) = \int_0^t d\tau\, \big[a_3(\tau) - a_2(\tau) g_1(\tau)\big]. \tag{11}$$

As a result, the operator $U(t)$ is thus determined.

## 3. CEV European options

The CEV model with time-dependent model parameters for a standard European option is described by the partial differential equation [5,6]

$$\frac{\partial P(S,\tau)}{\partial \tau} = \frac{1}{2}\sigma(\tau)^2 S^\beta \frac{\partial^2 P(S,\tau)}{\partial S^2} + \big[r(\tau) - d(\tau)\big] S \frac{\partial P(S,\tau)}{\partial S} - r(\tau) P(S,\tau) \tag{12}$$

for $0 \leqslant \beta < 2$. Here $P$ is the option value, $S$ is the underlying asset price, $\tau$ is the time to maturity, $\sigma$ is the volatility, $r$ is the risk-free interest rate, and $d$ is the dividend. Introducing a simple change of variables: $x = \sqrt{S^{(2-\beta)}}$, Eq. (12) becomes

$$\begin{aligned}
\frac{\partial u(x,\tau)}{\partial \tau} &= \frac{1}{8}\tilde{\sigma}(\tau)^2 \frac{\partial^2 u(x,\tau)}{\partial x^2} + \frac{1}{2}\left[\tilde{\mu}(\tau) x - \frac{(4-\beta)\tilde{\sigma}(\tau)^2}{4(2-\beta)x}\right] \frac{\partial u(x,\tau)}{\partial x} \\
&\quad + \left[\frac{(4-\beta)\tilde{\sigma}(\tau)^2}{8(2-\beta)x^2} - r(\tau) - \frac{\tilde{\mu}(\tau)}{2}\right] u(x,\tau) \\
&\equiv H(\tau) u(x,\tau),
\end{aligned} \tag{13}$$

where $\tilde{\sigma}(\tau) = (2-\beta)\sigma(\tau)$, $\tilde{\mu}(\tau) = (2-\beta)[r(\tau) - d(\tau)]$ and $u(x,\tau) = xP(S,\tau)$. It is not difficult to show that the operator $H(\tau)$ can be rewritten as follows:

$$H(\tau) = a_1(\tau) K_+ + a_2(\tau) K_0 + a_3(\tau) K_- + b(\tau), \tag{14}$$

where

$$\begin{aligned}
K_- &= \frac{1}{2}\left[\frac{\partial^2}{\partial x^2} - \frac{4-\beta}{(2-\beta)x}\frac{\partial}{\partial x} + \frac{4-\beta}{(2-\beta)x^2}\right], \\
K_0 &= \frac{1}{2}\left(x\frac{\partial}{\partial x} - \frac{1}{2-\beta}\right), \qquad K_+ = \frac{1}{2}x^2, \\
a_3(\tau) &= \frac{1}{4}\tilde{\sigma}(\tau)^2, \qquad a_2(\tau) = \tilde{\mu}(\tau), \\
a_1(\tau) &= 0, \qquad b(\tau) = -\frac{1-\beta}{2(2-\beta)}\tilde{\mu}(\tau) - r(\tau).
\end{aligned} \tag{15}$$

The operators $K_+$, $K_0$ and $K_-$ are the generators of the Lie algebra su(1, 1) [7]:

$$[K_+, K_-] = -2K_0, \qquad [K_0, K_\pm] = \pm K_\pm. \tag{16}$$

We may define the evolution operator $U(\tau, 0)$ such that

$$u(x,\tau) = \exp\left[\int_0^\tau d\tau'\, b(\tau')\right] U(\tau, 0) u(x, 0). \tag{17}$$

Inserting Eq. (17) into Eq. (13) gives the evolution equation

$$\frac{\partial}{\partial\tau}U(\tau,0)=H_I(\tau)U(\tau,0),\qquad U(0,0)=1 \tag{18}$$

with

$$H_I(\tau)=a_1(\tau)K_+ + a_2(\tau)K_0 + a_3(\tau)K_-. \tag{19}$$

Since the su(1, 1) algebra is a real "split 3-dimensional" simple Lie algebra, the Wei–Norman theorem states that the evolution operator $U(\tau,0)$ can be expressed in the form [3]

$$U(\tau,0)=\exp\bigl[c_1(\tau)K_+\bigr]\exp\bigl[c_2(\tau)K_0\bigr]\exp\bigl[c_3(\tau)K_-\bigr], \tag{20}$$

where the coefficients $c_i(\tau)$ are to be determined. Substituting Eqs. (19) and (20) into Eq. (18), we obtain, after direct differentiation and simplication [1]

$$c_1(\tau)=0, \tag{21}$$

$$c_2(\tau)=\int_0^\tau \tilde{\mu}(\tau')\,d\tau', \tag{22}$$

$$c_3(\tau)=\frac{1}{4}\int_0^\tau \tilde{\sigma}(\tau')^2\exp\bigl[c_2(\tau')\bigr]\,d\tau'. \tag{23}$$

Hence, we have found an exact form of the time evolution operator $U(\tau,0)$, which in turn gives the solution $u(x,\tau)$ of the pricing equation in Eq. (13).

## 4. Pricing formulae of moving barrier options

In this section we apply the results in Section 3 to derive the pricing formulae of the moving barrier options. First of all, we introduce the auxiliary function $\tilde{u}(x,\tau)$:

$$\begin{aligned}\tilde{u}(x,\tau)&\equiv\exp[\gamma K_+]u(x,\tau)\\&=\exp\Biggl[\int_0^\tau d\tau'\,b(\tau')\Biggr]\exp[\gamma K_+]\exp\bigl[c_2(\tau)K_0\bigr]\\&\quad\times\exp\bigl[c_3(\tau)K_-\bigr]\exp[-\gamma K_+]\tilde{u}(x,0),\end{aligned} \tag{24}$$

where $\gamma$ is a real adjustable parameter. Then we apply the Baker–Campbell–Hausdorff formula [7] to rewrite Eq. (24) as follows:

$$\begin{aligned}\tilde{u}(x,\tau)&=\exp\Biggl[\int_0^\tau d\tau'\,b(\tau')\Biggr]\exp[\gamma K_+]\exp\bigl\{c_3(\tau)\exp\bigl[-c_2(\tau)\bigr]K_-\bigr\}\\&\quad\times\exp\bigl[c_2(\tau)K_0\bigr]\exp[-\gamma K_+]\tilde{u}(x,0).\end{aligned} \tag{25}$$

Making use of the generalized normal- and antinormal-order decomposition formulae for the su(1, 1) algebra [8], the operator product in Eq. (25) can be rewritten in the normal-order form:

$$\begin{aligned}\tilde{u}(x,\tau) = {}& \exp\left[\int_0^\tau d\tau'\, b(\tau')\right]\exp[\gamma K_+]\exp\left\{-\frac{\gamma\exp[c_2(\tau)]}{1+\gamma c_3(\tau)}K_+\right\} \\ & \times \exp\left\{\left[c_2(\tau) - 2\ln\left|1+\gamma c_3(\tau)\right|\right]K_0\right\} \\ & \times \exp\left[\frac{c_3(\tau)}{1+\gamma c_3(\tau)}K_-\right]\tilde{u}(x,0). \end{aligned} \tag{26}$$

### 4.1. Up-and-out moving barrier options

Without loss of generality, we assume that $\tilde{u}(x,0) = x^{(\alpha+1)/2}v(x,0)$, where $\alpha = (4-\beta)/(2-\beta)$ and $v(x,0)$ is defined in terms of the Fourier–Bessel integral [9]:

$$v(x,0) = \sum_{n=1}^{\infty} \frac{2J_\omega(x_{\omega n}\frac{x}{L})}{L^2 J_{\omega+1}^2(x_{\omega n})}\int_0^L dy\, y J_\omega\left(x_{\omega n}\frac{y}{L}\right)v(y,0), \tag{27}$$

for $\omega \equiv (\alpha-1)/2 > -1$ and $0 \leqslant x \leqslant L$. Here $x_{\omega n}$ denotes the $n$th zero of the Bessel function $J_\omega$ of the first kind of order $\omega$. Then it is not difficult to show that

$$u(x,\tau) = \int_0^L dy\, K(x,\tau;y,0)u(y,0), \tag{28}$$

where

$$\begin{aligned} K(x,\tau;y,0) = {}& \sum_{n=1}^{\infty} \frac{2y}{L^2 J_{\omega+1}^2(x_{\omega n})}\left(\frac{x}{y}\right)^{\omega+1} \frac{\exp[c_2(\tau)/2 + \int_0^\tau d\tau' b(\tau')]}{|1+\gamma c_3(\tau)|} \\ & \times \exp\left\{-\frac{\gamma\exp[c_2(\tau)]}{2[1+\gamma c_3(\tau)]}x^2\right\}\exp\left\{-\frac{c_3(\tau)}{2[1+\gamma c_3(\tau)]L^2}x_{\omega n}^2\right\} \\ & \times J_\omega\left(x_{\omega n}\frac{\exp[c_2(\tau)/2]}{|1+\gamma c_3(\tau)|}\frac{x}{L}\right)J_\omega\left(x_{\omega n}\frac{y}{L}\right)\exp\left[\frac{1}{2}\gamma y^2\right]. \end{aligned} \tag{29}$$

In the above derivation we have made use of the fact that $x^{(\alpha+1)/2}J_{(\alpha-1)/2}(x\nu)$ is an eigenfunction of the operator $K_-$ with the eigenvalue $-\nu^2/2$ and the well-known relation

$$\exp\left(\eta x\frac{\partial}{\partial x}\right)f(x) = f\bigl(x\exp(\eta)\bigr). \tag{30}$$

It should be noted that at time $\tau \geqslant 0$ the kernel $K(x,\tau;y,0)$ vanishes at $x = L|1+\gamma c_3(\tau)| \times \exp[-c_2(\tau)/2]$. That is, we have derived the kernel of Eq. (13) with an absorbing barrier moving along the trajectory

$$x^*(\tau) = L\left|1+\gamma c_3(\tau)\right|\exp\bigl[-c_2(\tau)/2\bigr] \tag{31}$$

parametrized by the real adjustable parameter $\gamma$. As a result, the price of the corresponding **up-and-out** moving barrier option is given by

$$P_{\text{up-and-out}}(S,\tau) = \frac{u(x,\tau)}{x} = \frac{1}{x}\int_0^L dy\, K(x,\tau;y,0)u(y,0). \tag{32}$$

### 4.2. Down-and-out moving barrier options

On the other hand, if we suppose that $\tilde{u}(x,0) = x^{(\alpha+1)/2}\chi(x,0)$, where $\chi(x,0)$ is defined in terms of the Weber transform [10]:

$$\chi(x,0) = \int_0^\infty d\xi\, \frac{J_\omega(x\xi)Y_\omega(\xi L) - Y_\omega(x\xi)J_\omega(\xi L)}{J_\omega^2(\xi L) + Y_\omega^2(\xi L)}\xi \times \int_L^\infty dy\, \big[J_\omega(y\xi)Y_\omega(\xi L) - Y_\omega(y\xi)J_\omega(\xi L)\big] y\chi(y,0), \tag{33}$$

for $L \leqslant x < \infty$, then $u(x,\tau)$ is simply given by

$$u(x,\tau) = \int_L^\infty dy\, G(x,\tau;y,0)u(y,0), \tag{34}$$

where

$$\begin{aligned} G(x,\tau;y,0) = {} & \int_0^\infty d\xi\, y\xi \left(\frac{x}{y}\right)^{\omega+1} \frac{\exp[c_2(\tau)/2 + \int_0^\tau d\tau'\, b(\tau')]}{|1+\gamma c_3(\tau)|} \exp\left[\frac{1}{2}\gamma y^2\right] \\ & \times \exp\left\{-\frac{\gamma \exp[c_2(\tau)]}{2[1+\gamma c_3(\tau)]}x^2\right\} \exp\left\{-\frac{c_3(\tau)}{2[1+\gamma c_3(\tau)]}\xi^2\right\} \\ & \times \left[J_\omega\left(\frac{x\xi \exp[c_2(\tau)/2]}{|1+\gamma c_3(\tau)|}\right) Y_\omega(\xi L) - Y_\omega\left(\frac{x\xi \exp[c_2(\tau)/2]}{|1+\gamma c_3(\tau)|}\right) J_\omega(\xi L)\right] \\ & \times \frac{J_\omega(y\xi)Y_\omega(\xi L) - Y_\omega(y\xi)J_\omega(\xi L)}{J_\omega^2(\xi L) + Y_\omega^2(\xi L)} \end{aligned} \tag{35}$$

is the kernel of Eq. (13) associated with an absorbing barrier moving along the trajectory $x^*(\tau)$ given in Eq. (31). Here $Y_\omega$ denotes the Bessel function of the second kind of order $\omega$. In the above derivation we have made use of the fact that both $x^{(\alpha+1)/2}J_{(\alpha-1)/2}(x\nu)$ and $x^{(\alpha+1)/2}Y_{(\alpha-1)/2}(x\nu)$ are eigenfunctions of the operator $K_-$ with the eigenvalue $-\nu^2/2$. It should also be noted that the Gaussian decaying factor of the integrand ensures the rapid convergence of the integration over $\xi$. Accordingly, the price of the corresponding **down-and-out** moving barrier option is found to be

$$P_{\text{down-and-out}}(S,\tau) = \frac{u(x,\tau)}{x} = \frac{1}{x}\int_L^\infty dy\, G(x,\tau;y,0)u(y,0). \tag{36}$$

Furthermore, it is not difficult to see that in the special case of $L = 0$, i.e., **no barrier**, the kernel in Eq. (35) is reduced to the one obtained by Lo, Yuen and Hui [11], which has a Gaussian decaying factor in the variable $y$. By the *maximum principle* for the parabolic partial differential equation [4], we can thus conclude that the kernel in Eq. (35) must have a decaying factor in the variable $y$, which decays at least as fast as the Gaussian decaying factor in the special case of $L = 0$.

### 4.3. Illustrative applications

If we take a closer look at the trajectory of the moving barrier defined in Eq. (31), we would immediately realize that the special case of a **fixed barrier** does not belong to the class of parametric barriers. In order to simulate a fixed barrier, we shall thus choose an optimal value of the adjustable parameter $\gamma$ in such a way that the integral

$$\int_0^T \left[x^*(\tau) - L\right]^2 d\tau$$

is minimum. In other words, we try to minimize the deviation from the fixed barrier by varying the parameter $\gamma$. Here $T$ denotes the time at which the option price is evaluated. Within the framework of the new approach, we can also determine the upper and lower bounds for the exact barrier option prices. It is not difficult to show[1] that for an **up-and-out** option the upper bound can be provided by the option price associated with a moving barrier whose $x^*(\tau)$ is *greater than or equal to* $L$ for the duration of interest. Similarly, the option price associated with a moving barrier whose $x^*(\tau)$ is *less than or equal to* $L$ for the duration of interest can serve as the desired lower bound. In the example shown in Fig. 1, the best lower bound can be obtained by choosing

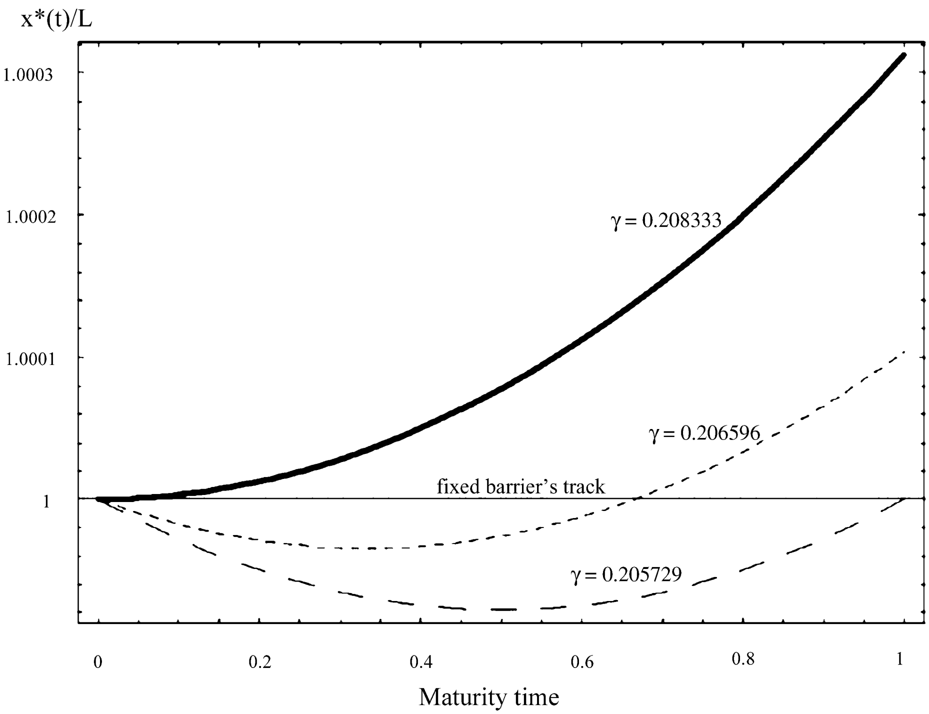


Fig. 1. Barrier tracks for the bounds and optimal estimate. The values of the parameter $\gamma$ are shown along the barrier tracks. Other input parameters are: $\beta = 1.0$, $S_0 = 26$, $\sigma_{BS}^2 = 0.02$, $d = 0$ and $r = 0.05$.

[1] The proof is based upon the maximum principle for the parabolic partial differential equation [4].

an appropriate value of $\gamma$ such that $x^*(\tau = 0) = x^*(\tau = T) = L$. That is, at time $\tau = T$ the moving barrier will return to its initial position and merge with the fixed barrier. On the other hand, the best upper bound can be obtained by choosing a $\gamma$ value which satisfies the requirement that $dx^*(\tau)/d\tau = 0$ at $\tau = 0$. That is, the instantaneous rate of change of $x^*(\tau)$ is required to be zero at time $\tau = 0$. On the contrary, for an **down-and-out** option we can simply switch the above two choices of barrier movement in order to determine the upper and lower bounds of the option price.

For illustration, we apply the approximate method to a "$\beta = 1$"-CEV **up-and-out** barrier call option with constant model parameters: $\sigma_{BS}^2 = 0.02$, $r = 0.05$, $d = 0$. (*Note* that the value of $\sigma$ to be used for the CEV model is adjusted to be $\sigma = \sigma_{BS} S^{(2-\beta)/2}$.) The strike price $X$ and the knockout barrier $S_0$ are set equal to 20 and 26, respectively. We now try to evaluate the barrier option price $P(S, \tau)$ associated with the current underlying asset price $S = 24$ at time $\tau = 1$.

First of all, we determine the optimal value of the adjustable parameter $\gamma$:

$$\gamma_{\text{opt}} = 0.206596. \tag{37}$$

Then an estimate of the exact up-and-out barrier option price can be evaluated by numerically computing the integral in Eq. (32) (with, for example, *Mathematica*):

$$P(S = 24, \tau = 1) = 0.71396. \tag{38}$$

Since the exact value of the barrier option price is found to be [12]

$$P_{\text{exact}}(S = 24, \tau = 1) = 0.71401, \tag{39}$$

the approximate estimate is indeed very close to the exact result with a percentage error of $-0.007\%$ only. The numerical results for the corresponding upper and lower bounds are determined as follows:

$$\begin{aligned} &\text{Upper bound} = 0.71641 \quad (\%\ \text{error} = 0.34\%;\ \gamma = 0.208333) \\ &\text{Lower bound} = 0.71274 \quad (\%\ \text{error} = -0.18\%;\ \gamma = 0.205729). \end{aligned} \tag{40}$$

Clearly, the new approach is able to yield very tight upper and lower bounds for the exact barrier option price.

In order to assess the efficiency of the new approach, we also perform Monte Carlo simulation to evaluate the option price. Using a time-step of $10^{-5}$ and a sample of $10^5$ random paths of the underlying asset price, the Monte Carlo method needs about an hour to compute one estimate on an *AXP 900MHz Workstation*, while the new approach consumes less than a minute on a *Pentium III 667MHz PC* to determine one estimate using *Mathematica*. Moreover, the Monte Carlo method gives a much poorer estimate, namely 0.73651 with a percentage error of 3.2%, in comparison with the new approach.

## 5. Conclusion

In this paper we have applied the Lie-algebraic technique to derive the analytical kernels of the pricing formulae of both up-and-out and down-and-out moving barrier options in the CEV model environment. The moving barriers are parametrized by some real adjustable parameters as shown in Eq. (31). With these analytical kernels, we are able to compute the prices of the moving barrier options and the associated hedge parameters very efficiently. In view of the CEV model being empirically considered to be a better candidate in equity option pricing than the traditional

Black–Scholes model because the CEV process allows for a nonzero elasticity of return variance with respect to prices [13–16]. Our new approach could facilitate more efficient comparative pricing and precise risk management in equity derivatives with barriers by incorporating term-structures of interest rates, volatility and dividend into the CEV option valuation model.

In addition to providing a better description of stock behaviour, the CEV process can be employed in the contingent-claims approach to valuing defaultable bonds. For example, in a valuation model of defaultable bonds proposed by Cathcart and El-Jahel [17] recently, default occurs when some signaling process hits some constant default barrier (that is, the option to default can be considered as a barrier option). The model assumes the signaling process for each firm that determines the occurrence of default rather than the value of the assets of the firm. The signaling process can capture factors that can affect the probability of default. The use of the signaling process is also appropriate for entities such as sovereign issuers that issue defaultable debts but do not have an identifiable collection of assets [18]. The signaling process could follow diffusion processes such as lognormal, Ornstein–Uhlenbeck, or CEV processes.

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